STAT 441 - Mathematical Statistics I
Due: Friday (at the start of class), Oct. 20, 2017

Instructions: *Include all relevant work to get full credit.*

Homework 14

1. Do problem 5.89 on page 268.
2. Do problem 5.92 on page 269.
3. Do problem 5.103 on page 276.
4. Do all parts of problem 5.114 on page 278. [*Note that both density functions are Gamma density functions.*]

Solutions:

#1

$y_2 \backslash y_1$	0	1	2	$p(y_2)$
0	1/9	2/9	1/9	4/9
1	2/9	2/9	0	4/9
2	1/9	0	0	1/9
$p(y_1)$	4/9	4/9	1/9	

$$\Rightarrow E(Y_2) = 0\left(\tfrac{4}{9}\right) + 1\left(\tfrac{4}{9}\right) + 2\left(\tfrac{1}{9}\right) = \tfrac{6}{9} = \tfrac{2}{3}$$

$$E(Y_1) = 0\left(\tfrac{4}{9}\right) + 1\left(\tfrac{4}{9}\right) + 2\left(\tfrac{1}{9}\right) = \tfrac{6}{9} = \tfrac{2}{3}$$

$$E(Y_1Y_2) = (0)(0)\left(\tfrac{1}{9}\right) + (0)(1)\left(\tfrac{2}{9}\right) + (0)(2)\left(\tfrac{1}{9}\right) + (1)(0)\left(\tfrac{2}{9}\right) + (1)(1)\left(\tfrac{2}{9}\right) + (1)(2)(0) + (2)(0)\left(\tfrac{1}{9}\right) + (2)(1)(0) + (2)(2)(0) = \tfrac{2}{9}$$

$$\Rightarrow Cov(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = \tfrac{2}{9} - \tfrac{2}{3}\left(\tfrac{2}{3}\right) = \tfrac{2}{9} - \tfrac{4}{9} = -\tfrac{2}{9}$$

The covariance is negative because as Y_1 increases, Y_2 tends to decrease.

#2.

$$f(y_1, y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

From homework #13, we know that $E(Y_1) = \frac{1}{4}$ and $E(Y_2) = \frac{1}{2}$.

$$E(Y_1Y_2) = \int_0^1 \int_{y_1}^1 y_1 y_2 6(1-y_2)\, dy_2\, dy_1 = \int_0^1 6y_1\left(\tfrac{1}{2}y_2^2 - \tfrac{1}{3}y_2^3\right)\Big|_{y_2=y_1}^{y_2=1} dy_1$$

$$= \int_0^1 6y_1\left(\tfrac{1}{6} - \tfrac{1}{2}y_1^2 + \tfrac{1}{3}y_1^3\right) dy_1 = \int_0^1 (y_1 - 3y_1^3 + 2y_1^4)\, dy_1$$

$$= \left(\frac{y_1^2}{2} - \frac{3y_1^4}{4} + \frac{2}{5}y_1^5\right)\Big|_0^1 = \frac{1}{2} - \frac{3}{4} + \frac{2}{5} = \frac{10-15+8}{20} = \frac{3}{20}$$

$$\Rightarrow Cov(Y_1, Y_2) = \frac{3}{20} - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{3}{20} - \frac{1}{8} = \frac{6-5}{40} = \frac{1}{40}$$

Since the $Cov(Y_1, Y_2) \neq 0$, then Y_1 + Y_2 are not independent.

#3 Given: $E(Y_1) = 2 \quad , E(Y_2) = -1 \quad , E(Y_3) = 4$

$V(Y_1) = 4 \quad , V(Y_2) = 6 \quad , V(Y_3) = 8$

$\text{cov}(Y_1, Y_2) = 1 \quad , \text{Cov}(Y_1, Y_3) = -1 \, , \text{Cov}(Y_2, Y_3) = 0$

a) $E(3Y_1 + 4Y_2 - 6Y_3) = 3E(Y_1) + 4E(Y_2) - 6E(Y_3)$ by Theorem ~~5.11~~ 5.12

$= 3(2) + 4(-1) - 6(4)$

$= 6 - 4 - 24 = -22$

b) $V(3Y_1 + 4Y_2 - 6Y_3) = 3^2 V(Y_1) + 4^2 V(Y_2) + (-6)^2 V(Y_3)$

$+ 2\Big((3)(4)\,\text{Cov}(Y_1, Y_2) + (3)(-6)\,\text{Cov}(Y_1, Y_3) + 4(-6)\,\text{Cov}(Y_2, Y_3)\Big)$

$= 9(4) + 16(6) + 36(8) + 2\big(12(1) - 18(-1) - 24(0)\big)$

$= 36 + 96 + 288 + 2(12 + 18) = 480$

#4 Given: $f_1(y_1) = \begin{cases} \frac{1}{6} y_1^3 e^{-y_1} & , y_1 > 0 \\ 0 & , \text{elsewhere} \end{cases}$ and $f_2(y_2) = \begin{cases} \frac{1}{2} e^{-y_2/2} & , y_2 > 0 \\ 0 & , y_2 \leq 0 \end{cases}$

Note: $Y_1 \sim \text{Gamma}(\alpha = 4, \beta = 1)$ and $Y_2 \sim \text{Gamma}(\alpha = 1, \beta = 2)$

$\Rightarrow E(Y_1) = \alpha\beta = 4(1) = 4$ and $E(Y_2) = \alpha\beta = 2$

$V(Y_1) = \alpha\beta^2 = 4$ + $V(Y_2) = \alpha\beta^2 = 4$

a) $E(U) = E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = 4 - 2 = 2.$

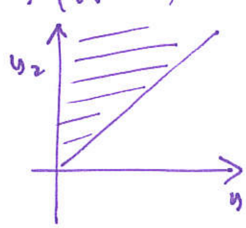
b) $V(U) = V(Y_1 - Y_2) = V(Y_1) + V(Y_2) + 2(1)(-1)\,\text{Cov}(Y_1, Y_2)$

$= V(Y_1) + V(Y_2) - 2\,\text{Cov}(Y_1, Y_2)$, but $\text{cov}(Y_1, Y_2) = 0$ since $Y_1 + Y_2$ are independent.

$= V(Y_1) + V(Y_2)$

$= 4 + 4 = 8$

c) $P(U < 0) = P(Y_1 - Y_2 < 0) = P(Y_1 < Y_2) = \int_0^\infty \int_{y_1}^\infty \left(\frac{1}{6} y_1^3 e^{-y_1}\right)\left(\frac{1}{2} e^{-y_2/2}\right) dy_2\, dy_1$



$= \int_0^\infty \frac{1}{6} y_1^3 e^{-y_1} \left(-e^{-y_2/2}\Big|_{y_1}^\infty\right) dy_1 = \int_0^\infty \frac{1}{6} y_1^3 e^{-y_1} \left(e^{-y_1/2}\right) dy_1$

$= \frac{1}{6} \int_0^\infty \underbrace{y_1^3 e^{-y/(2/3)}}_{\text{Kernel of Gamma } (\alpha = 4, \beta = 2/3)} dy_1 = \frac{1}{6} \left(\frac{2}{3}\right)^4 \Gamma(4) = \left(\frac{2}{3}\right)^4 \approx .1975$

$\Rightarrow$ No, only about 20% of the time.