STAT 441 - Mathematical Statistics I
Due: Monday (at the start of class), Oct. 30, 2017

Instructions: *Include all relevant work to get full credit.*

Homework 15

1. Do problem 5.126 on page 283.
2. Do all parts of problem 5.133 on page 289.
3. Do all parts of problem 5.136 on page 289.
4. Do all parts of problem 6.1 on page 307.

Solutions:

1) $(Y_1, Y_2, Y_3) \sim \text{Multinomial}(n=10, p_1 = .10, p_2 = .05, p_3 = .85)$

$\Rightarrow E(Y_1) = np_1 = 10(.10) = 1$, $E(Y_2) = np_2 = 10(.05) = .5$

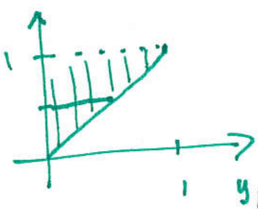
$V(Y_1) = np_1(1-p_1) = .9$, $V(Y_2) = np_2(1-p_2) = .5(.95) \approx .475$

$\Rightarrow Cov(Y_1, Y_2) = -np_1p_2 = -10(.1)(.05) = -.05$

a) $E(Y_1 + 3Y_2) = E(Y_1) + 3E(Y_2) = 1 + 3(.5) = 2.5$

$V(Y_1+3Y_2) = V(Y_1) + 9V(Y_2) + 2(3)Cov(Y_1,Y_2) = .9 + 9(.475) + 6(-.05) = 4.875$

2) $f(y_1, y_2) = \begin{cases} 6(1-y_2) & , 0 \le y_1 \le y_2 \le 1 \\ 0 & , \text{elsewhere} \end{cases}$



$\Rightarrow f_{Y_2}(y_2) = \int_0^{y_2} 6(1-y_2)\,dy_1 = 6(1-y_2)y_1 \Big|_0^{y_2} = 6y_2(1-y_2)$, $0 \le y_2 \le 1$

$\Rightarrow f_{1|2}(y_1|y_2) = \frac{f(y_1,y_2)}{f_2(y_2)} = \frac{\cancel{6}\cancel{(1-y_2)}}{\cancel{6}y_2\cancel{(1-y_2)}} = \frac{1}{y_2}$, $0 \le y_1 \le y_2$

a) $\Rightarrow E(Y_1|Y_2) = \int_0^{y_2} y_1 f(y_1|y_2)\,dy_1 = \int_0^{y_2} y_1\left(\frac{1}{y_2}\right)dy_1 = \frac{1}{y_2}\frac{y_1^2}{2}\Big|_0^{y_2} = \frac{1}{2}y_2$

b) $E(Y_1) = E(E(Y_1|Y_2)) = E\left(\frac{1}{2}Y_2\right) = \int_0^1 \frac{1}{2}y_2\,(6y_2(1-y_2))\,dy_2$

$= \int_0^1 (3y_2^2 - 3y_2^3)\,dy_2 = \left(y_2^3 - \frac{3}{4}y_2^4\right)\Big|_0^1 = 1 - \frac{3}{4} = \frac{1}{4}$

3) a) $E(Y) = E(E(Y|\lambda)) = E(\lambda) = \int_0^\infty \lambda e^{-\lambda}\,d\lambda = \Gamma(2) = 1.$

b) $\cancel{E}V(Y) = E[V(Y|\lambda)] + V[E(Y|\lambda)] = E[\lambda] + V[\lambda] = 1 + 1 = 2.$

Note: $f_{Y|\lambda}(y|\lambda) = \text{poisson}(\lambda) \Rightarrow E(Y|\lambda) = \lambda$ + $V(Y|\lambda) = \lambda.$

and $\lambda \sim \exp(\beta = 1) \Rightarrow E(\lambda) = 1$ + $V(\lambda) = 1.$

c) Since $E(Y)=1$ + $SD(Y)=\sqrt{2}\approx 1.414$, it is very unlikely that $Y>9$.

OR, you can compute $P(Y>9)=1-P(Y\le 9)$.

Note, $Y|\lambda \sim$ Poisson $(\lambda) \Rightarrow p(y|\lambda)=\frac{\lambda^y e^{-\lambda}}{y!}$, $y=0,1,2,\dots$

$\lambda \sim Exp(\beta=1) \Rightarrow f(\lambda)=e^{-\lambda}$, $\lambda\ge 0$

$$\Rightarrow f(y,\lambda)=f(y|\lambda)f(\lambda)=\frac{\lambda^y e^{-\lambda}}{y!}(e^{-\lambda})=\frac{\lambda^y e^{-2\lambda}}{y!}$$

$$\Rightarrow f_Y(y)=\int_0^\infty \frac{\lambda^y e^{-\lambda/(\frac{1}{2})}}{y!}d\lambda=\frac{1}{y!}\Gamma(y+1)\left(\tfrac{1}{2}\right)^{y+1}=\left(\tfrac{1}{2}\right)^{y+1},\quad y=0,1,\dots$$

$$\Rightarrow P(Y>9)=1-P(Y\le 9)=1-\sum_{y=0}^{9}\left(\tfrac{1}{2}\right)^{y+1}=1-\left[\tfrac{1}{2}+\left(\tfrac{1}{2}\right)^2+\dots+\left(\tfrac{1}{2}\right)^{10}\right]=1-\left[1-\left(\tfrac{1}{2}\right)^{10}\right]$$

$$=\left(\tfrac{1}{2}\right)^{10}\approx .000977$$

4) $f_Y(y)=2(1-y)$, $0\le y\le 1$

a) $$f_U(u)=\frac{d}{du}P(U\le u)=\frac{d}{du}P\left(\cancel{2Y-1}\le u\right)=\frac{d}{du}P\left(Y\le \tfrac{1}{2}(u+1)\right)=\frac{d}{du}F_Y\left(\tfrac{1}{2}(u+1)\right)$$

$$=f_Y\left(\tfrac{1}{2}(u+1)\right)\cdot\left(\tfrac{1}{2}\right)=2\left(1-\tfrac{1}{2}(u+1)\right)\left(\tfrac{1}{2}\right)=1-\tfrac{1}{2}u-\tfrac{1}{2}=\tfrac{1}{2}-\tfrac{1}{2}u,\quad -1\le u\le 1.$$

Since $U=2Y-1$ and $0\le y\le 1 \Rightarrow -1\le 2y-1\le 1$

b) $U=1-2Y$ and $0\le y\le 1 \Rightarrow -2\le -2y\le 0 \Rightarrow 1-2\le u\le 1 \Rightarrow -1\le u\le 1$

$$f_U(u)=\frac{d}{du}F(u)=\frac{d}{du}Pr(U\le u)=\frac{d}{du}P(1-2Y\le u)=\frac{d}{du}P\left(Y\ge -\tfrac{1}{2}(u-1)\right)=\frac{d}{du}\left[1-F_Y\left(\tfrac{1}{2}(1-u)\right)\right]$$

$$=0-f_Y\left(\tfrac{1}{2}(1-u)\right)\left(-\tfrac{1}{2}\right)=\tfrac{1}{2}\left[2\left(1-\tfrac{1}{2}(1-u)\right)\right]$$

$$=1-\tfrac{1}{2}+\tfrac{1}{2}u=\tfrac{1}{2}+\tfrac{1}{2}u,\quad -1\le u\le 1$$

c) $U=Y^2 \Rightarrow 0\le u\le 1$.

$$f_U(u)=\frac{d}{du}F(u)=\frac{d}{du}Pr(Y^2\le u)=\frac{d}{du}Pr(-\sqrt{u}\le Y\le\sqrt{u})=\frac{d}{du}\left[F(\sqrt{u})-\overset{F(0)}{\cancel{F(-\sqrt{u})}}\right]$$

$$=f_Y(\sqrt{u})\left(\frac{1}{2\sqrt{u}}\right)=\frac{1}{2\sqrt{u}}\left[f_Y(\sqrt{u})\right]$$

$$=\frac{1}{2\sqrt{u}}\left[2(1-\sqrt{u})\right]=\frac{1-\sqrt{u}}{\sqrt{u}}=\frac{1}{\sqrt{u}}-1,\quad 0\le u\le 1.$$

d) $$E(U_1)=\int_{-1}^{1}u_1\left(\tfrac{1}{2}-\tfrac{1}{2}u_1\right)du_1=\left(\tfrac{1}{2}\tfrac{u_1^2}{2}-\tfrac{1}{2}\tfrac{u_1^3}{3}\right)\Big|_{-1}^{1}=\left(\tfrac{1}{4}-\tfrac{1}{6}\right)-\left(\tfrac{1}{4}+\tfrac{1}{6}\right)=\tfrac{1}{12}-\tfrac{5}{12}=-\tfrac{4}{12}=-\tfrac{1}{3}$$

chapter 4: $$E(U_1)=E(2Y-1)=2E(Y)-1=2\int_0^1 y\,2(1-y)\,dy-1=2\left(\tfrac{2y^2}{2}-\tfrac{2y^3}{3}\right)\Big|_0^1-1=\tfrac{2}{3}-1=-\tfrac{1}{3}$$

$$E(U_2)=\int_{-1}^{1}u_2\left(\tfrac{1}{2}+\tfrac{1}{2}u_2\right)du=\left(\tfrac{u_2^2}{4}+\tfrac{u_2^3}{6}\right)\Big|_{-1}^{1}=\left(\tfrac{1}{4}+\tfrac{1}{6}\right)-\left(\tfrac{1}{4}-\tfrac{1}{6}\right)=\tfrac{5}{12}-\tfrac{1}{12}=\tfrac{4}{12}=\tfrac{1}{3}$$

$$E(U_3)=\int_0^1 u_3\left(\tfrac{1}{\sqrt{u_3}}-1\right)du_3=\int_0^1\left(u_3^{1/2}-u_3\right)du_3=\tfrac{2}{3}u_3^{3/2}-\tfrac{u_3^2}{2}\Big|_0^1=\tfrac{2}{3}-\tfrac{1}{2}=\tfrac{1}{6}$$

Chapter 4: $$E(Y^2)=\int_0^1 y^2(2(1-y))\,dy=\left(\tfrac{2y^3}{3}-\tfrac{2y^4}{4}\right)\Big|_0^1=\tfrac{2}{3}-\tfrac{1}{2}=\tfrac{1}{6}.$$