STAT 441 - Mathematical Statistics I
Due: Friday (at the start of class), Nov. 3, 2017

Instructions: *Include all relevant work to get full credit.*

Homework 16

1. Do part (a) of problem 6.26 on page 317.
2. Do part (a) of problem 6.27 on page 317.
3. Do all parts of problem 6.34 on page 318.
4. Do all parts (a) and (b) of problem 6.43 on page 323.
5. Do problem 6.45 on page 323.
6. Do part (a) of problem 6.52 on page 324.

Solutions:

1) $f(y) = \frac{1}{\alpha} m y^{m-1} e^{-y^m/\alpha}$, $y>0$. If $U = Y^m$ (this is a monotonic function)

$\Rightarrow Y = \sqrt[m]{u} \rightarrow \frac{dy}{du} = \frac{1}{m} u^{\frac{1}{m}-1}$

$\Rightarrow f_U(u) = f_Y(y)\left|\frac{dy}{du}\right| = \frac{1}{\alpha} m (\sqrt[m]{u})^{m-1} e^{-u/\alpha}\left|\frac{dy}{du}\right|$, $u>0$

$= \frac{\cancel{m}}{\alpha} \cancel{(u)^{\frac{m-1}{m}}} e^{-u/\alpha} \frac{1}{\cancel{m}} \cancel{u^{\frac{1-m}{m}}} = \frac{1}{\alpha} e^{-u/\alpha}$, $u>0$

$\Rightarrow U \sim \text{Exp}(\text{mean} = \alpha)$

2) Given $f(y) = \frac{1}{\beta} e^{-y/\beta}$, $y>0$ and $W = \sqrt{Y} \Rightarrow Y = W^2 \Rightarrow \frac{dy}{dw} = 2w$

$\Rightarrow f_W(w) = f_Y(y)\left|\frac{dy}{dw}\right| = \frac{1}{\beta} e^{-w^2/\beta}|2w|$, $w>0$.

$= \frac{1}{\beta} 2 w^{2-1} e^{-w^2/\beta} \sim \text{Weibull}(\alpha = \beta, m = 2)$

3) Given $f(y) = \left(\frac{2y}{\theta}\right) e^{-y^2/\theta}$, $y>0$

a) If $U = Y^2 \Rightarrow Y = \sqrt{u}$ and $\frac{dy}{du} = \frac{1}{2\sqrt{u}}$

$\Rightarrow f_U(u) = f_Y(y)\left|\frac{dy}{du}\right| = \left(\frac{\cancel{2\sqrt{u}}}{\theta}\right) e^{-u/\theta}\left|\frac{1}{\cancel{2\sqrt{u}}}\right|$, $u>0$.

$= \frac{1}{\theta} e^{-u/\theta}$, $u>0$ $\Rightarrow U \sim \text{Exp}(\text{mean} = \theta)$.
$\Rightarrow E(U) = \theta$ + $V(U) = \theta^2$

b) $E(Y) = E(\sqrt{U}) = \int_0^\infty \frac{u^{1/2}}{\theta} e^{-u/\theta} du$
$= \frac{1}{\theta}\int_0^\infty u^{\frac{3}{2}-1} e^{-u/\theta} du = \frac{1}{\theta} \cdot \Gamma\left(\frac{3}{2}\right)\theta^{3/2} = \sqrt{\theta} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\theta}\sqrt{\pi}}{2}$

c) $V(Y) = E(Y^2) - (E(Y))^2 = E(U) - \left(\frac{\sqrt{\theta}\sqrt{\pi}}{2}\right)^2 = \theta - \frac{1}{4}\theta\pi = \theta\left(\frac{4-\pi}{4}\right)$

4) (a) $\bar{Y} = \frac{1}{n}\sum_1^n Y_i \overset{\text{by Theorem 6.3}}{\Longrightarrow} \bar{Y} \sim \text{Normal}\left(\mu_{\bar{Y}} = \frac{1}{n}\sum_1^n \mu = \mu,\ \sigma^2_{\bar{Y}} = \left(\frac{1}{n}\right)^2 \sum_1^n \sigma^2 = \frac{\sigma^2}{n}\right)$

(b) $P(|\bar{Y} - \mu| \le 1) = P(-1 \le \bar{Y} - \mu \le 1) = P\left(\frac{-\sqrt{n}}{\sigma} \le \frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} \le \frac{\sqrt{n}}{\sigma}\right)$

$= \Phi\left(\frac{\sqrt{n}}{\sigma}\right) - \Phi\left(\frac{-\sqrt{n}}{\sigma}\right) = \Phi\left(\frac{5}{4}\right) - \Phi\left(-\frac{5}{4}\right) = .8944 - .1056 \doteq .7888$

5) $U = 100 + 7Y_1 + 3Y_2$

Since $Y_1 \sim N(\mu_1 = 10,\ \sigma_1 = .5)$ and $Y_2 \sim N(\mu_2 = 4,\ \sigma_2 = .2)$

$\Rightarrow U \sim \text{Normal}\left(\mu_U = 100 + 7(10) + 3(4) = 182,\ \sigma^2_U = 7^2(.5)^2 + 3^2(.2)^2 = 12.61\right)$

Let b = bid amount

$\Rightarrow P(U > b) = .01 \Rightarrow P\Big(\underbrace{\frac{U - \mu_U}{\sigma_U}}_{=Z} > \frac{b - 182}{\sqrt{12.61}}\Big) = .01$

$\Rightarrow \frac{b-182}{\sqrt{12.61}} \approx Z_{.01} = 2.326$

$\Rightarrow b = 182 + 2.326\sqrt{12.61} \approx 190.26$

b) Yes, it is reasonable to assume that Y_1 & Y_2 are independent, because

6) Given $Y_1 \sim \text{Poisson}(\lambda_1)$ & $Y_2 \sim \text{Poisson}(\lambda_2)$

$\Rightarrow m_{Y_1}(t) = e^{\lambda_1(e^t - 1)}$ & $m_{Y_2}(t) = e^{\lambda_2(e^t-1)}$

If $U = Y_1 + Y_2 \Rightarrow m_U(t) = e^{\lambda_1(e^t-1)} * e^{\lambda_2(e^t-1)}$

$= e^{(\lambda_1+\lambda_2)(e^t-1)}$

$\Rightarrow U \sim \text{Poisson}(\text{mean} = \lambda_1 + \lambda_2)$