STAT 441 - MATHEMATICAL STATISTICS I
DUE: WEDNESDAY (AT THE START OF CLASS), NOV. 8, 2017

Instructions: *Include all relevant work to get full credit.*

Homework 17

1. Do problem 6.31 on page 317.
2. Do problem 6.36 on page 318.
3. Do all parts of problem 6.64 on page 331.

Solutions:

1) $U = \frac{Y_2}{Y_1} \Rightarrow Y_2 = UY_1 \Rightarrow \frac{dy_2}{du} = y_1$

$$\Rightarrow g(u, y_1) = \frac{1}{8} y_1 e^{-(y_1 + uy_1)/2} |y_1| \quad , \quad y_1 > 0 \ \& \ u > 0$$

$$= \frac{1}{8} y_1^2 e^{-y(1+u)/2}$$

$$\Rightarrow f_U(u) = \int_0^\infty \frac{1}{8} y_1^{3-1} e^{-y/\left(\frac{2}{1+u}\right)} dy_1 = \frac{1}{8} \Gamma(3) \left(\frac{2}{1+u}\right)^3 = \frac{2}{(1+u)^3} \quad , \quad u > 0.$$

2) $U = Y_1^2 + Y_2^2 \Rightarrow Y_2 = \sqrt{u - Y_1^2} \Rightarrow \frac{dy_2}{du} = \frac{1}{2\sqrt{u - y_1^2}}$

$$f_{Y_1,Y_2}(y_1, y_2) = \left(\frac{2y_1}{\theta}\right) e^{-y_1^2/\theta} \left(\frac{2y_2}{\theta}\right) e^{-y_2^2/\theta} = \frac{4y_1 y_2}{\theta^2} e^{-\frac{1}{\theta}(y_1^2 + y_2^2)}$$

$$\Rightarrow g(u, y_1) = \frac{4y_1 \cancel{\sqrt{u - y_1^2}}}{\theta^2} e^{-\frac{u}{\theta}} \frac{1}{2\cancel{\sqrt{u - y_1^2}}} = \frac{2}{\theta^2} y_1 e^{-u/\theta} \quad , \quad 0 < y_1 < \sqrt{u}$$

$$\Rightarrow f_U(u) = \int_0^{\sqrt{u}} \frac{2}{\theta^2} y_1 e^{-u/\theta} dy_1 = \frac{1}{\theta^2} y_1^2 e^{-u/\theta} \Big|_0^{\sqrt{u}} = \frac{u}{\theta^2} e^{-u/\theta} \quad , \quad u > 0.$$

$$\Rightarrow U \sim Gamma(\alpha = 2, \beta = \theta)$$

3) $U_1 = \frac{Y_1}{Y_1 + Y_2}$ and $U_2 = Y_1 + Y_2 \Rightarrow Y_1 = U_1 U_2$ and $Y_2 = U_2 - U_1 U_2$

$$\Rightarrow J = \begin{vmatrix} u_2 & u_1 \\ -u_2 & 1 - u_1 \end{vmatrix} = u_2(1 - u_1) + u_1 u_2 = u_2$$

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{y_1^{\alpha_1 - 1} e^{-y_1/\beta}}{\Gamma(\alpha_1)\beta^{\alpha_1}} \cdot \frac{y_2^{\alpha_2 - 1} e^{-y_2/\beta}}{\Gamma(\alpha_2)\beta^{\alpha_2}} \quad , \quad y_1 > 0 \ , \ y_2 > 0$$

(a) $$\Rightarrow f_{U_1,U_2}(u_1, u_2) = \frac{(u_1 u_2)^{\alpha_1 - 1} (u_2 - u_1 u_2)^{\alpha_2 - 1} e^{-\frac{1}{\beta}(u_1 u_2 + u_2 - u_1 u_2)}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1 + \alpha_2}} (u_2) \quad , \quad 0 < u_1 < 1 \ , \ u_2 > 0$$

$$= \frac{u_2 (u_2)^{\alpha_1 - 1} u_2^{\alpha_2 - 1} u_1^{\alpha_1 - 1} (1 - u_1)^{\alpha_2 - 1} e^{-u_2/\beta}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1 + \alpha_2}} = \frac{u_2^{\alpha_1 + \alpha_2 - 1} u_1^{\alpha_1 - 1} (1 - u_1)^{\alpha_2 - 1} e^{-u_2/\beta}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1 + \alpha_2}}$$

(b)
$$f_{U_1}(u_1) = \int_0^\infty \frac{u_1^{\alpha_1-1}(1-u_1)^{\alpha_2-1}\, u_2^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} e^{-u_2/\beta}\, du_2$$

$$= \frac{u_1^{\alpha_1-1}(1-u_1)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cancel{\beta^{\alpha_1+\alpha_2}}} \underbrace{\int_0^\infty u_2^{\alpha_1+\alpha_2-1} e^{-u_2/\beta}\, du_2}_{=\Gamma(\alpha_1+\alpha_2)\cancel{\beta^{\alpha_1+\alpha_2}}}$$

$$= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u_1^{\alpha_1-1}(1-u_1)^{\alpha_2-1} \quad , 0 < u_1 < 1$$

$\Rightarrow U_1 \sim \text{Beta}(\alpha_1, \alpha_2)$

(c)
$$f_{U_2}(u_2) = \frac{u_2^{\alpha_1+\alpha_2-1} e^{-u_2/\beta}}{\cancel{\Gamma(\alpha_1)}\cancel{\Gamma(\alpha_2)}\beta^{\alpha_1+\alpha_2}} \underbrace{\int_0^1 u_1^{\alpha_1-1}(1-u_1)^{\alpha_2-1}\, du_1}_{=\frac{\cancel{\Gamma(\alpha_1)}\cancel{\Gamma(\alpha_2)}}{\Gamma(\alpha_1+\alpha_2)}}$$

$$= \frac{u_2^{\alpha_1+\alpha_2-1} e^{-u_2/\beta}}{\Gamma(\alpha_1+\alpha_2)\beta^{\alpha_1+\alpha_2}} \quad , \quad u_2 > 0$$

$\Rightarrow U_2 \sim \text{Gamma}(\alpha = \alpha_1+\alpha_2, \beta)$

(d)
Note that $f_{U_1,U_2}(u_1,u_2) = f_{U_1}(u_1) * f_{U_2}(u_2)$ from parts (a), (b), & (c).

Therefore, U_1 & U_2 are independent.