STAT 441 - Mathematical Statistics I
Due: Friday (at the start of class), Nov. 10, 2017

Instructions: *Include all relevant work to get full credit.*

Homework 18

1. Do all parts of problem 6.80 on page 339.
2. Do problem 6.84 on page 339.
3. Do all parts of problem 6.87 on page 339.
4. Do all parts of problem 6.88 on page 340.

Solutions:

1) $Y_1, Y_2, \dots, Y_n \overset{iid}{\sim} \text{Beta}(\alpha=2, \beta=2) \Rightarrow f_Y(y) = \frac{\Gamma(4)}{\Gamma(2)\Gamma(2)} y(1-y), \quad 0<y<1$

$= 6y(1-y)$

$\Rightarrow F_Y(y) = \int_0^y 6t(1-t)\,dt$

$= 3t^2 - 2t^3 \Big|_0^y = 3y^2 - 2y^3$

(a) $\Rightarrow F_{Y_{(n)}} = [F_Y(y)]^n = (3y^2-2y^3)^n$

(b) $f_{Y_{(n)}}(y) = n(3y^2-2y^3)^{n-1}(6y-6y^2), \quad 0<y<1$

(c) $E(Y_{(n)}) = \int_0^1 y\, 2(3y^2-2y^3)(6y-6y^2)\,dy = \int_0^1 12(y^4)(3-2y)(1-y)\,dy$

$= \int_0^1 12(3y^4 - 5y^5 + 2y^6)\,dy = 12\left(\frac{3}{5}y^5 - \frac{5}{6}y^6 + \frac{2}{7}y^7\right)\Big|_0^1$

$= 12\left(\frac{3}{5} - \frac{5}{6} + \frac{2}{7}\right) \approx 0.6286 = \frac{22}{35}$

2) $f(y) = \frac{1}{\alpha} m y^{m-1} e^{-y^m/\alpha}, \quad y>0$

$\Rightarrow F_Y(y) = \int_0^y \frac{m}{\alpha} t^{m-1} e^{-t^m/\alpha}\,dt = \int_0^{-\frac{y^m}{\alpha}} -e^u\,du = -e^u \Big|_0^{-\frac{y^m}{\alpha}} = 1 - e^{-y^m/\alpha}$

let $u = -\frac{t^m}{\alpha} \Rightarrow \frac{du}{dt} = -\frac{mt^{m-1}}{\alpha}$

$\Rightarrow F_{Y_{(1)}}(y) = 1 - [1 - F_Y(y)]^n = 1 - \left[e^{-\frac{y^m}{\alpha}}\right]^n = 1 - e^{-\frac{ny^m}{\alpha}}$

$\Rightarrow f_{Y_{(1)}}(y) = +\frac{nmy^{m-1}}{\alpha} e^{-\frac{ny^m}{\alpha}} \sim \text{Weibull}\left(\frac{\alpha}{n}, m\right)$

3) $f_Y(y) = \frac{1}{2} e^{-\frac{1}{2}(y-4)}$, $y \geq 4$

$\Rightarrow F_Y(y) = \int_4^y \frac{1}{2} e^{-\frac{1}{2}(t-4)} dt = -e^{-\frac{1}{2}(t-4)}\Big|_4^y = 1 - e^{-\frac{1}{2}(y-4)}$

(a) $\Rightarrow f_{Y_{(1)}}(y) = n\left[1-F_Y(y)\right]^{n-1} f_Y(y) = \cancel{2}\left(e^{-\frac{1}{2}(y-4)}\right)\frac{1}{\cancel{2}} e^{-\frac{1}{2}(y-4)} = e^{-(y-4)}$, $y \geq 4$

(b) $E(Y_{(1)}) = \int_4^\infty y e^{-y+4} dy = -ye^{-y+4}\Big|_4^\infty - \int_4^\infty -e^{-y+4} dy$

Let $u = y$ + $dv = e^{-y+4} dy$

$du = dy$ $\quad v = -e^{-y+4} + C$

$\overset{L'H}{=} (0+4) + \left(-e^{-y+4}\Big|_4^\infty\right) = 4 + (0+1) = 5$

4) $f_Y(y) = e^{-(y-\theta)}$, $y > \theta$

$\Rightarrow F_Y(y) = \int_\theta^y e^{-(t-\theta)} dt = -e^{-(t-\theta)}\Big|_\theta^y = 1 - e^{-(y-\theta)}$, $y > \theta$

(a) $\Rightarrow f_{Y_{(1)}}(y) = n\left[1-F_Y(y)\right]^{n-1} f_Y(y) = n\left(e^{-(y-\theta)}\right)^{n-1} e^{-(y-\theta)}$, $y > \theta$

$= \begin{cases} n e^{-n(y-\theta)} & , y > \theta \\ 0 & , \text{elsewhere} \end{cases}$

(b) $E(Y_{(1)}) = \int_\theta^\infty y n e^{-n(y-\theta)} dy$

Let $u = y$ $\quad dv = ne^{-n(y-\theta)} dy$

$du = dy$ $\quad v = -e^{-n(y-\theta)}$

$= -ye^{-n(y-\theta)}\Big|_\theta^\infty + \int_\theta^\infty e^{-n(y-\theta)} dy$

$\overset{L'H}{=} (0+\theta) + \left(-\frac{1}{n} e^{-n(y-\theta)}\Big|_\theta^\infty\right)$

$= \theta + \left(0 + \frac{1}{n}\right)$

$= \theta + \frac{1}{n}$