STAT 441 - Mathematical Statistics I
Due: Monday (at the start of class), Dec. 4, 2017

Instructions: *Include all relevant work to get full credit.*

Homework 20

1. Do all parts of problem 8.6 on page 394.
2. Do all parts of problem 8.12 on page 395.
3. Do all parts of problem 8.13 on page 395.
4. Do problem 8.18 on page 396.

Solutions:

1) Given: $E(\hat\theta_1) = E(\hat\theta_2) = \theta$, $V(\hat\theta_1) = \sigma_1^2$, and $V(\hat\theta_2) = \sigma_2^2$

(a) $E(\hat\theta_3) = E(a\hat\theta_1 + (1-a)\hat\theta_2) = aE(\hat\theta) + (1-a)E(\hat\theta_2) = a\theta + (1-a)\theta = \theta$

(b) $V(\hat\theta_3) = a^2 V(\hat\theta_i) + (1-a)^2 V(\hat\theta_2) = a^2\sigma_1^2 + (1-a)^2\sigma_2^2 = a^2\sigma_1^2 + (1-2a+a^2)\sigma_2^2 = a^2(\sigma_1^2+\sigma_2^2) - 2a\sigma_2^2 + \sigma_2^2$

$\Rightarrow \frac{dV(\hat\theta_3)}{da} = 2a(\sigma_1^2+\sigma_2^2) - 2\sigma_2^2 = 0 \Rightarrow a = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$

Since $\frac{d^2V(\hat\theta_3)}{da^2} = 2(\sigma_1^2+\sigma_2^2) > 0 \Rightarrow V(\hat\theta_3)$ is minimun at $a = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$

2) $Y_i \overset{iid}{\sim} Unif(\theta, \theta+1) \Rightarrow E(Y_i) = \theta + \frac{1}{2}$ + $V(Y_i) = \frac{1}{12}$

(a) $E(\bar Y) = E(\frac{1}{n}\sum Y_i) = \frac{1}{n}\sum E(Y_i) = \frac{1}{n}[n(\theta+\frac{1}{2})] = \theta + \frac{1}{2} \neq \theta. \Rightarrow B(\hat\theta) = \frac{1}{2}$

(b) $\hat\theta = \bar Y - \frac{1}{2} \Rightarrow E(\hat\theta) = E(\bar Y) - \frac{1}{2} = \theta$

(c) $MSE(\bar Y) = V(\bar Y) + [B(\bar Y)]^2 = \frac{1/12}{n} + (\frac{1}{2})^2 = \frac{1}{12n} + \frac{1}{4}$

3) $Y \sim Bin(n,p) \Rightarrow E(Y) = np$ + $V(Y) = np(1-p)$

(a) $E(\hat\theta) = E(n\hat p(1-\hat p)) = nE(\hat p - \hat p^2) = nE(\hat p) - nE(\hat p^2) = np - n\left[\frac{p(1-p)}{n} + p^2\right]$

where, $\hat p = \frac{Y}{n} \Rightarrow E(\hat p) = E(\frac{Y}{n}) = \frac{1}{n}E(Y) = \frac{1}{n}(np) = p$

$\Rightarrow V(\hat p) = V(\frac{Y}{n}) = (\frac{1}{n})^2 V(Y) = \frac{1}{n^2}(np(1-p)) = \frac{p(1-p)}{n}$

$\Rightarrow E(\hat p^2) = V(\hat p) + (E(\hat p))^2 = \frac{p(1-p)}{n} + p^2$

$= np - p(1-p) - np^2$

$= np(1-p) - p(1-p)$

$= (n-1)p(1-p)$

$\neq V(Y)$

(b) Let $\hat\theta^* = \frac{n}{n-1}\hat\theta = \frac{n^2}{n-1}\hat p(1-\hat p)$

4) Given: $Y_i \overset{iid}{\sim} Unif[0,\theta] \Rightarrow F_Y(y) = \frac{y}{\theta}$, $0<y<\theta$

Let $u = \frac{y}{\theta}$

$du = \frac{1}{\theta}dy$

$\Rightarrow F_{Y_{(1)}}(y) = 1 - [1-\frac{y}{\theta}]^n \Rightarrow f_{Y_{(1)}}(y) = -n(1-\frac{y}{\theta})^{n-1}(-\frac{1}{\theta})$

$\Rightarrow E(Y_{(1)}) = E(\hat\theta) = \int_0^\theta y\left[-n(1-\frac{y}{\theta})^{n-1}(-\frac{1}{\theta})\right]dy = \int_0^\theta n(\frac{y}{\theta})(1-\frac{y}{\theta})^{n-1}dy = n\int_0^1 u(1-u)^{n-1}\theta\, du$

$= n\theta\int_0^1 u(1-u)^{n-1}du = n\theta\left(\frac{\Gamma(2)\Gamma(n)}{\Gamma(n+2)}\right) = n\theta\left(\frac{1}{n(n+1)}\right) = \frac{\theta}{n+1}$

$\Rightarrow \hat\theta^* = (n+1)Y_{(1)} \rightarrow E(\hat\theta^*) = \theta.$