STAT 441 - MATHEMATICAL STATISTICS I
DUE: WEDNESDAY (AT THE START OF CLASS), DEC. 6, 2017

Instructions: *Include all relevant work to get full credit.*

Homework 21

1. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample taken from a normal population with mean μ and variance σ^2. If $\bar{Y}$ and S^2 represent the sample mean and sample variance, respectively, we have shown in class that the quantity $T = \dfrac{\bar{Y} - \mu}{S/\sqrt{n}}$ has a $t-$distribution with $(n-1)$ degrees of freedom.

a. Show that the $(1-\alpha)100\%$ confidence interval for μ is given by $\bar{Y} \pm t_{\alpha/2}\left(\dfrac{S}{\sqrt{n}}\right)$, where $t_{\alpha/2}$ is the value of T such that $P(T > t_{\alpha/2}) = \alpha/2$.

b. Do problem 8.80 on page 430. Provide an interpretation of the confidence interval you obtained in the context of the problem.

2. Given two independent random samples: $X_1, X_2, \ldots, X_{n_1}$ from $N(\mu_1, \sigma^2)$ and $Y_1, Y_2, \ldots, Y_{n_2}$ from $N(\mu_2, \sigma^2)$. Let $\bar{X}$ and S_1^2 represent the sample mean and sample variance, respectively, of sample 1, and $\bar{Y}$ and S_2^2 represent the sample mean and sample variance, respectively, of sample 2.

a. Use definition 7.2 to show that $T = \dfrac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_p^2}{n_1} + \frac{S_p^2}{n_2}}} = \dfrac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$ follows the $t-$distribution with $(n_1 + n_2 - 2)$ degrees of freedom, where $S_p^2 = \dfrac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$. [Hint: Use Theorem 7.2].

b. Use the result of part (a) to show that the $(1-\alpha)100\%$ confidence interval for $\mu_1 - \mu_2$ is given by

$$(\bar{X} - \bar{Y}) \pm t_{\alpha/2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}},$$

where $t_{\alpha/2}$ is the value of T such that $P(T > t_{\alpha/2}) = \alpha/2$.

c. Use the result from part (b) to do all parts of problem 8.90 on page 433.

Solutions:

1) (a) Since $T = \dfrac{\bar{Y} - \mu}{S/\sqrt{n}} \sim t_{df=n-1}$

$\Rightarrow Pr(-t_{\alpha/2} < T < t_{\alpha/2}) = 1-\alpha$

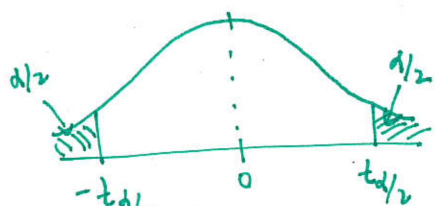
$\Rightarrow Pr\left(-t_{\alpha/2} < \dfrac{\bar{Y} - \mu}{S/\sqrt{n}} < t_{\alpha/2}\right) = 1-\alpha$

$\Rightarrow Pr\left(-t_{\alpha/2}\dfrac{S}{\sqrt{n}} < \bar{Y} - \mu < t_{\alpha/2}\dfrac{S}{\sqrt{n}}\right) = 1-\alpha$

$\Rightarrow Pr\left(-\bar{Y} - t_{\alpha/2}\dfrac{S}{\sqrt{n}} < -\mu < -\bar{Y} + t_{\alpha/2}\dfrac{S}{\sqrt{n}}\right) = 1-\alpha$

$\Rightarrow Pr\left(\bar{Y} + t_{\alpha/2}\dfrac{S}{\sqrt{n}} > \mu > \bar{Y} - t_{\alpha/2}\dfrac{S}{\sqrt{n}}\right) = 1-\alpha$

$\Rightarrow Pr\left(\bar{Y} - t_{\frac{\alpha}{2}}\dfrac{S}{\sqrt{n}} < \mu < \bar{Y} + t_{\frac{\alpha}{2}}\dfrac{S}{\sqrt{n}}\right) = 1-\alpha$



Hence, the $(1-\alpha)100\%$ confidence interval for μ is $\left[\bar{Y} - t_{\frac{\alpha}{2}}\dfrac{S}{\sqrt{n}},\ \bar{Y} + t_{\frac{\alpha}{2}}\dfrac{S}{\sqrt{n}}\right]$

(b) Given: $n = 21$, $\bar{y} = 26.6$, $s = 7.4$

$\Rightarrow \bar{y} \pm t_{\frac{\alpha}{2}}\dfrac{s}{\sqrt{n}} = 26.6 \pm 2.086\left(\dfrac{7.4}{\sqrt{21}}\right) = 26.6 \pm 3.37 = [23.23, 29.97]$

We are 95% confident that the true mean post treatment self-esteem sure (μ) is between 23.23 and 29.97.

2) $X_1, X_2, \ldots, X_{n_1} \overset{iid}{\sim} N(\mu_1, \sigma^2) \overset{\text{Thm 7.1}}{\Rightarrow} \bar{X} \sim N\left(\mu_{\bar{X}} = \mu_1, \sigma^2_{\bar{X}} = \frac{\sigma^2}{n_1}\right)$

(a) $Y_1, Y_2, \ldots, Y_{n_2} \overset{iid}{\sim} N(\mu_2, \sigma^2) \Rightarrow \bar{Y} \sim N\left(\mu_{\bar{Y}} = \mu_2, \sigma^2_{\bar{Y}} = \frac{\sigma^2}{n_2}\right)$

By Theorem 6.3, $\bar{X} - \bar{Y} \sim N\left(\mu_{\bar{X}-\bar{Y}} = \mu_1 - \mu_2, \sigma^2_{\bar{X}-\bar{Y}} = \frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}\right)$

$$\Rightarrow Z = \frac{(\bar{X}-\bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}}} = \frac{(\bar{X}-\bar{Y}) - (\mu_1 - \mu_2)}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim N(0,1)$$

Also, by Theorem 7.3

$$\frac{(n_1-1)S_1^2}{\sigma^2} \sim \chi^2_{df=n_1-1} \quad + \quad \frac{(n_2-1)S_2^2}{\sigma^2} \sim \chi^2_{df=n_2-1}$$

$$\Rightarrow W = \frac{(n_1-1)S_1^2}{\sigma^2} + \frac{(n_2-1)S_2^2}{\sigma^2} \sim \chi^2_{df=n_1+n_2-2}$$

By Definition 7.2,

$$T = \frac{Z}{\sqrt{W/\nu}} = \frac{\dfrac{(\bar{X}-\bar{Y}) - (\mu_1-\mu_2)}{\cancel{\sigma}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}}{\sqrt{\dfrac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{\cancel{\sigma^2}}\Big/(n_1+n_2-2)}} = \frac{(\bar{X}-\bar{Y}) - (\mu_1-\mu_2)}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \sim t_{df=(n_1+n_2-2)}$$

(b) Therefore, $P_r\left(-t_{\alpha/2} < T < t_{\alpha/2}\right) = 1-\alpha$

$$\Rightarrow P_r\left(-t_{\alpha/2} < \frac{(\bar{X}-\bar{Y}) - (\mu_1-\mu_2)}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} < t_{\alpha/2}\right) = 1-\alpha$$

$$\Rightarrow P_r\left((\bar{X}-\bar{Y}) - t_{\alpha/2} S_p\sqrt{\tfrac{1}{n_1}+\tfrac{1}{n_2}} < (\mu_1-\mu_2) < (\bar{X}-\bar{Y}) + t_{\frac{\alpha}{2}} S_p\sqrt{\tfrac{1}{n_1}+\tfrac{1}{n_2}}\right) = 1-\alpha$$

Hence, the $(1-\alpha)100\%$ confidence interval for $(\mu_1-\mu_2)$ is given by

$$(\bar{X}-\bar{Y}) \pm t_{\frac{\alpha}{2}} S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$$

(c) [a] $(534-446) \pm 2.048\sqrt{\frac{14(42^2)+14(45^2)}{28}}\sqrt{\frac{1}{15}+\frac{1}{15}} = 88 \pm 2.048(43.5)\sqrt{\frac{2}{15}}$

$= 88 \pm 32.53 = [55.47, 120.53]$

[b] $(517-548) \pm 2.048\,(54.56)\sqrt{\frac{2}{15}}$

$= -31 \pm 40.08 = [-71.08, 9.08]$

[c] With 95% confidence level, $(\mu_2 - \mu_E)$ for verbal is between 55.47 + 120.53 and also with 95% confidence level, $(\mu_2 - \mu_E)$ for Math is between -71.08 and 9.08

[d] The samples are independent, normal, + Equal SD for each test.