STAT 441 - MATHEMATICAL STATISTICS I
DUE: MONDAY (AT THE START OF CLASS), DEC. 11, 2017

Instructions: *Include all relevant work to get full credit.*

Homework 22 (Last Homework!!)

1. Do all parts of problem 8.44 on page 410.
2. Do all parts of problem 8.48 on page 411.
3. Do all parts of problem 8.132 on page 442.

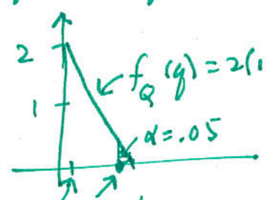
Solutions:

1) (a) $F(y) = P(Y \le y) = \int_0^y \frac{2(\theta - t)}{\theta^2}\,dt = \frac{2\theta t - t^2}{\theta^2}\Big|_0^y = \frac{2\theta y}{\theta^2} - \frac{y^2}{\theta^2} = \frac{2y}{\theta} - \frac{y^2}{\theta^2}, \quad 0 < y < \theta$

when $y \le 0 \Rightarrow F(y) = P(Y \le 0) = 0$ + when $y > \theta \Rightarrow F(y) = P(Y \le \theta) = \frac{2\theta}{\theta} - \frac{\theta^2}{\theta^2} = 1$

(b) First, note that $Q = \frac{Y}{\theta}$ is a function of only Y and θ.
Also, $f_Q(q) = f_Y(y)\left|\frac{dy}{dq}\right| = \frac{2(\theta - \theta q)}{\theta^2}|\theta| = 2(1-q), \quad 0 < q < 1$ since $y = \theta q$ + $\frac{dy}{dq} = \theta$

which is free of θ.



(c) $\Rightarrow \int_0^{q_{.95}} 2(1-q)\,dq = .05$

$\Rightarrow -(1-q)^2\Big|_0^{q_{.95}} = 1 - (1-q_{.95})^2 = .05$

$\Rightarrow (1-q_{.95})^2 = .95$

$\Rightarrow q_{.95} = 1 - \sqrt{.95} \approx .025$

$\Rightarrow \frac{1}{2}(1-q_{.05})(2-2q_{.05}) = .05$

or $\int_{q_{.05}}^1 2(1-q)\,dq = .05$

$\Rightarrow (1-q)^2 = .05 \Rightarrow q = 1 - \sqrt{.05} \approx .776$

$\Rightarrow P(.025 \le Q \le .776) \approx .90$

$\Rightarrow .025 \le \frac{Y}{\theta} \le .776$

$\Rightarrow \frac{1}{.776} \le \frac{\theta}{Y} \le \frac{1}{.025}$

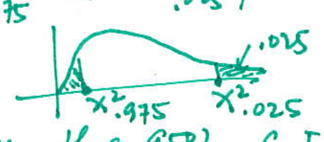
$\Rightarrow P\left(\frac{Y}{.776} \le \theta \le \frac{Y}{.025}\right) = .90 \quad \Rightarrow$ Hence, the 90% C.I. for θ is given by $\left[\frac{Y}{.776}, \frac{Y}{.025}\right]$

2) $Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim} G(\alpha = 2, \beta) \Rightarrow m_{Y_i}(t) = (1-\beta t)^{-2} \Rightarrow m_{\frac{2Y_i}{\beta}}(t) = m_{Y_i}\left(\frac{2t}{\beta}\right) = (1-2t)^{-2}$

(a) Note that $Q = \frac{2\sum Y_i}{\beta}$ is a function of the sample values + β only, and its distribution is free of β because if $W = \frac{2Y_i}{\beta}$ and $Q = \sum_{i=1}^n W_i$

$m_Q(t) = m_{W_1}(t)\,m_{W_2}(t)\cdots m_{W_n}(t) = (1-2t)^{-2}(1-2t)^{-2}\cdots(1-2t)^{-2} = (1-2t)^{-2n}$. $\sim \chi^2_{df=4n}$

(b) $P(\chi^2_{.975} \le Q \le \chi^2_{.025}) = .95 \Rightarrow P\left(\chi^2_{.975} \le \frac{2\sum Y_i}{\beta} \le \chi^2_{.025}\right) = .95$



$\Rightarrow P\left(\frac{2\sum Y_i}{\chi^2_{.025}} \le \beta \le \frac{2\sum Y_i}{\chi^2_{.975}}\right) = .95$

Hence, the 95% C.I. for β is given by $\left[2\sum Y_i / \chi^2_{.025},\ 2\sum Y_i / \chi^2_{.975}\right]$

(c) Given: $n = 5$, $\bar{y} = 5.39$ $\Rightarrow$ The 95% C.I. for β is

$\hookrightarrow df = 4n = 20$

$$\left[\frac{2(5 * 5.39)}{34.1696}, \frac{2(5 * 5.39)}{9.5908} \right]$$

$$= [\ 1.58\ ,\ 5.62\ ]$$

3)

(a) $F_{Y_{(n)}}(y) = [F_Y(y)]^n = \left[\left(\frac{y}{\theta}\right)^{\alpha}\right]^n = \left(\frac{y}{\theta}\right)^{n\alpha} = \left(\frac{y}{\theta}\right)^{nc}$, when $\alpha = c$.

(b) Note that $Q = \frac{Y_{(n)}}{\theta}$ is a function of the sample values and θ.

Also, $F_Q(q) = P(Q \le q) = P\left(\frac{Y_{(n)}}{\theta} \le q\right) = P(Y_{(n)} \le \theta q) = \left[\left(\frac{\theta q}{\theta}\right)^{nc}\right] = q^{nc}$

$\Rightarrow f_Q(q) = nc(q)^{nc-1}$, $0 < q < 1$.

This distribution is free of θ.

Also, for k in $(0,1)$

$$P_r\left(k < \frac{Y_{(n)}}{\theta} < 1\right) = \int_k^1 nc\, q^{nc-1} dq = q^{nc}\Big|_k^1 = 1 - k^{nc}.$$

(c) (i) $P\left(k < \frac{Y_{(5)}}{\theta} \le 1\right) = .95 \Rightarrow 1 - k^{5(2.4)} = .95$

$\Rightarrow k^{12} = .05 \Rightarrow k = \sqrt[12]{.05} \approx .779$

(ii) The 95% one-sided confidence interval for θ

is given by $\left[\frac{Y_{(5)}}{1}, \frac{Y_{(5)}}{k}\right]$

$$= \left[Y_{(5)}, \frac{Y_{(5)}}{.779}\right]$$