STAT 441 - MATHEMATICAL STATISTICS I
DUE: FRIDAY (AT THE START OF CLASS), SEPT. 22, 2017

Instructions: *Include all relevant work to get full credit.*

Homework 5

1. A particular concentration of a chemical found in polluted water has been found to be lethal to 40% of the fish that are exposed to the concentration for 24 hours. Twenty five (25) fish are placed in a tank containing this concentration of chemical in water for 24 hours. [3]

   a. Let X denote the number of fish that survive out of the 25 fish placed in the tank. Give the name and the parameter values of the distribution of X. $X \sim \text{Binomial}(n = 25, p = .6)$

   b. Find the probability that exactly 14 survive. *Use 4 decimal places.*

   $P(X=14) = \binom{25}{14}(.6)^{14}(.4)^{11} \approx 0.1465$

   c. Find the probability that at most 14 survive. *Use 4 decimal places.*

   $P(X \leq 14) = F(14) \approx 0.4142$

   d. Find the probability that at least 12 survive. *Use 4 decimal places.*

   $P(X \geq 12) = 1 - F(11) = 1 - .0778 = .9222$

   e. Find the mean and variance of the number that survive.

   $\mu = np = 25(.6) = 15$ and $V(x) = \sigma^2 = np(1-p) = 25(.6)(.4) = 6$

2. Suppose there are n trials in a binomial experiment and we observe y_0 "successes," show that $P(Y = y_0)$ is maximized when $p = y_0/n$. *Make sure to check that you get a maximum and not a minimum at this point.*

   [Hint: The maximum of $P(Y = y_0)$ and $\ln(P(Y = y_0))$ occur at the same place]. [3]

   Let $L = \ln(P(Y=y_0)) = \ln\left[\binom{n}{y_0} p^{y_0}(1-p)^{n-y_0}\right] = \ln\binom{n}{y_0} + y_0 \ln p + (n-y_0)\ln(1-p)$

   $\Rightarrow \frac{dL}{dp} = 0 + \frac{y_0}{p} - \frac{n-y_0}{1-p} = 0 \Rightarrow \frac{y_0}{p} = \frac{n-y_0}{1-p} \Rightarrow y_0 - \cancel{y_0 p} = np - \cancel{y_0 p} \Rightarrow p = y_0/n$

   $\Rightarrow \frac{d^2L}{dp^2} = -\frac{y_0}{p^2} - \frac{(n-y_0)}{(1-p)^2} < 0$ since y_0, $(n-y_0)$, $+p^2$ are nonnegative. $(1-p)^2$ Therefore, $P(Y=y_0)$ is maximum at $p = \frac{y_0}{n}$

3. The *maximum likelihood estimator* for p is $\hat{p} = Y/n$, where $Y \sim$ binomial(n, p). Derive $E(\hat{p})$ and $V(\hat{p})$. [2]

   a) $E(\hat{p}) = E\left(\frac{Y}{n}\right) = \frac{1}{n}E(Y) = \frac{1}{n}(np) = p$ [So $\hat{p}$ is an unbiased estimator of p]

   b) $V(\hat{p}) = V\left(\frac{1}{n}\cdot Y\right) = \left(\frac{1}{n}\right)^2 V(Y) = \frac{1}{n^2}(np(1-p)) = \frac{p(1-p)}{n}$

4. If X has a geometric distribution with success probability p,

   a. show that $P(X = \text{an odd integer}) = \dfrac{p}{1-(1-p)^2}$. [1]

   $$P(X = \text{an odd integer}) = P(X=1) + P(X=3) + P(X=5) + \cdots$$
   $$= p + p(1-p)^2 + p(1-p)^4 + \cdots$$
   $$\left(\frac{a_1}{1-r}\right) = \frac{p}{1-(1-p)^2} \blacksquare$$

   b. show that $E(\hat{p}) = \dfrac{-p\ln(p)}{1-p}$, where $\hat{p} = \frac{1}{X}$ is the *maximum likelihood estimator* for p.

   [Hint: If $|r| < 1, \sum_{i=1}^{\infty} \frac{r^i}{i} = -\ln(1-r)$.] [1]

   $$E(\hat{p}) = E\left(\frac{1}{X}\right) = \sum_{x=1}^{\infty} \frac{1}{x} p(1-p)^{x-1} = \frac{p}{1-p}\sum_{x=1}^{\infty}\frac{1}{x}(1-p)^x = \left(\frac{p}{1-p}\right)\left(-\ln(1-(1-p))\right)$$
   $$= \frac{-p\ln(p)}{1-p} \blacksquare$$