STAT 441 - Mathematical Statistics I
Due: Friday (at the start of class), Sept. 29, 2017

Instructions: *Include all relevant work to get full credit.*

Homework 7

1. Suppose X has a binomial distribution with n trials and probability of success p.

a. Show that the moment-generating function for X is

$$m(t) = (pe^t + q)^n, \qquad \text{where } q = 1 - p.$$

[Hint: Think of the binomial expansion $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$]

b. Use the result from part (a) and Theorem 3.12 to show that the mean of X is np.

c. Use the results from parts (a), (b), and Theorem 3.12 to show that the variance of X is $np(1-p)$.

2. If Y has a geometric distribution with probability of success p, show that the moment-generating function for Y is

$$m(t) = \frac{pe^t}{1 - qe^t}, \qquad \text{where } q = 1 - p.$$

[Recall: The sum of an infinite geometric series is $\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$ when $|r| < 1$.]

3. Use the uniqueness of moment-generating functions to determine the exact distribution (*give the name and parameter values*) of the random variables that have each of the following moment-generating functions:

a. $m(t) = e^{5(e^t - 1)}$ $\rightarrow$ Poisson $(\lambda = 5)$

b. $m(t) = [(1/5)e^t + (4/5)]^{10}$ $\rightarrow$ Binomial $(n=10, p=\frac{1}{5})$

c. $m(t) = \dfrac{2e^t}{5 - 3e^t} = \dfrac{(2/5)e^t}{1-(3/5)e^t}$ $\rightarrow$ Geometric $(p = \frac{2}{5})$

Solutions:

1) $X \sim Bin(n, p) \Rightarrow P(X=x) = \binom{n}{x} p^x q^{n-x}$, $x = 0, 1, \dots, n$

(a) $m(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{\infty} \binom{n}{x} (pe^t)^x q^{n-x} = (pe^t + q)^n$ ∎

(b) $\mu_X = \left.\dfrac{d\,m(t)}{dt}\right|_{t=0} = \left. n(pe^t + q)^{n-1}(pe^t)\right|_{t=0} = n\underbrace{(pe^0 + q)^{n-1}}_{= p+q = 1}(pe^0) = np$ ∎

(c) $\mu_2' = \left.\dfrac{d^2 m(t)}{dt^2}\right|_{t=0} = \left. n(n-1)(pe^t + q)^{n-2}(pe^t)(pe^t)\right|_{t=0} + \left. n(pe^t+q)^{n-1}(pe^t)\right|_{t=0}$

$= (n^2 - n)(p+q)^{n-2} p^2 + n(p+q)^{n-1}(p) = n^2p^2 - np^2 + np$

$\Rightarrow V(X) = \mu_2' - (\mu_1')^2 = (n^2p^2 - np^2 + np) - (np)^2 = -np^2 + np = np(1-p)$ ∎

2) $Y \sim Geo(p) \Rightarrow P(Y=y) = p(1-p)^{y-1}$, $y = 1, 2, 3, \dots$

$\Rightarrow m(t) = E(e^{tY}) = \sum_{y=1}^{\infty} e^{ty} p(q)^{y-1} = \frac{p}{q} \sum_{y=1}^{\infty} (e^t)^y q^y = \frac{p}{q} \sum_{y=1}^{\infty} (qe^t)^y$

$= \frac{p}{q} \cdot \frac{qe^t}{1 - qe^t} = \frac{pe^t}{1 - qe^t}$ ∎