STAT 441 - Mathematical Statistics I
Due: Friday (at the start of class), Oct. 6, 2017

Instructions: *Include all relevant work to get full credit.*

Homework 8

1. A gas station operates two pumps, each of which can pump up to 10,000 gallons of gas in a month. The total amount of gas pumped at the station in a month is a random variable Y (measured in 10,000 gallons) with a probability density function given by

$$f(y) = \begin{cases} y, & 0 \le y < 1, \\ 2-y, & 1 \le y \le 2, \\ 0 & \text{elsewhere.} \end{cases}$$

a. Graph $f(y)$.

b. Find $F(y)$ and graph it.

c. Find the probability that the station will pump between 8000 and 12,000 gallons in a particular month.

d. Given that the station pumped more than 10,000 gallons in a particular month, find the probability that the station pumped more than 15,000 gallons during the month.

e. Use definition 4.5, $E(Y) = \int_{-\infty}^{\infty} y f(y)\,dy$, to show that $E(Y) = 1$.

[Hint: Split the integral into 2 parts – one for $0 \le y < 1$ and another for $1 \le y \le 2$.]

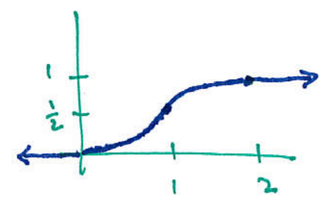
f. Use the fact that $V(Y) = E(Y^2) - (E(Y))^2$ to show that $V(Y) = \frac{1}{6}$.

g. Calculate the probability that Y will be within 2 standard deviations from the mean. Does it satisfy the Tchebysheff's Theorem?

Solutions:

a)

b) $F(y) = \begin{cases} 0 & , y < 0 \\ \frac{1}{2}y^2 & , 0 \le y < 1 \\ -\frac{y^2}{2} + 2y - 1 & , 1 \le y \le 2 \\ 1 & , y > 2 \end{cases}$



c) $P(.8 < Y < 1.2) = F(1.2) - F(.8) = \left(-\frac{(1.2)^2}{2} + 2(1.2) - 1\right) - \frac{1}{2}(.8)^2 \approx .68 - .32 = .36$

OR by symmetry $= 1 - 2P(Y < .8) = 1 - 2(.32) = .36$

d) $P(Y > 1.5 \mid Y > 1) = \dfrac{P(Y > 1.5 \cap Y > 1)}{P(Y > 1)} = \dfrac{P(Y > 1.5)}{\frac{1}{2}} \underset{\text{by symmetry}}{=} \dfrac{P(Y < .5)}{\frac{1}{2}} = \dfrac{\frac{1}{2}(.5)^2}{1/2} = .25$

e) $E(Y) = \int_0^1 y^2 dy + \int_1^2 y(2-y)\,dy = \frac{y^3}{3}\Big|_0^1 + \left(y^2 - \frac{y^3}{3}\right)\Big|_1^2 = \frac{1}{3} + \left(\frac{4}{3} - \frac{2}{3}\right) = 1.$

f) $E(Y^2) = \int_0^1 y^3 dy + \int_1^2 y^2(2-y)\,dy = \frac{y^4}{4}\Big|_0^1 + \left(\frac{2y^3}{3} - \frac{y^4}{4}\right)\Big|_1^2 = \frac{1}{4} + \left(\frac{16}{12} - \frac{5}{12}\right) = \frac{14}{12}$

$\Rightarrow V(Y) = E(Y^2) - (E(Y))^2 = \frac{14}{12} - 1^2 = \frac{2}{12} = \frac{1}{6}$. $\Rightarrow SD(Y) \approx .408$

g) $P(\mu - 2\sigma < Y < \mu + 2\sigma) = P(1 - 2(.408) < Y < 1 + 2(.408)) = P(.184 < Y < 1.816)$

$= F(1.816) - F(.184) = 1 - 2 * P(Y < .184)$

$= 1 - 2\left(\frac{1}{2}(.184)^2\right) \approx .966$

Yes, it satisfies Tchebysheff's Theorem.