Probability and Counting Techniques

- Definitions
  - *Experiment* - is the process by which an observation is made.
  - *Sample Space* - Set of all possible outcomes.
  - *Event* - A subset of the sample space.
  - *Probability (of an event)*, $P(E)$ - The chance of this event occurring. It is equal to the sum of the sample point probabilities in the event.

- **Equally Likely Model.** If an experiment has n different possible outcomes each of which has the same chance of occurring, then the probability that a specific outcome will occur is $1/n$. If an event occurs in k out of these n outcomes, then the probability of the event E, $P(E)$, is

$$P(E) = \frac{\text{no. of favorable outcomes}}{\text{no. of possible outcomes}} = \frac{k}{n}$$

  Examples:

  1. If a card is drawn from a well-shuffled standard deck of cards, what is the probability of getting

  a. a heart?

  b. a face card?

  c. a heart or a face card?

  2. If two cards are drawn, what is the probability of getting a pair (two of a kind)?

  3. If five cards are drawn, what is the probability of getting a full house (3 of a kind and a pair)?

- **Fundamental Principle of Counting (Product Rule) [Theorem 2.1].** Suppose two tasks are to be performed in succession. If the first task can be done in exactly n_1 different ways, and the second task can be done independently in n_2 different ways, then the sequence of things can be done in $n_1 \times n_2$ different ways.

  Examples:

  1. If you roll a six-sided die and then pick a card from the standard deck of cards, how many different possible outcomes are there?

  2. In a certain town, there are 3 male and 2 female candidates for mayor and 4 male and 2 female candidates for vice-mayor. If each candidate has the same chance of winning the election, what is the probability that after the election they will have a female mayor and a male vice-mayor?

  3. Do number 2.27 on page 39.

- **General Fundamental Principle of Counting .** Suppose k tasks are to be performed in succession. If the first task can be done in exactly n_1 different ways, the second task can be done independently in n_2 different ways, the third task can be done independently in n_3 different ways, and so forth, then this sequence of k tasks can be done in $n_1 n_2 n_3 \cdots n_k$ different ways.

  Examples:

  1. In a statistical study, an individual is classified according to gender, income bracket (upper, middle, or lower class) and highest level of educational attained (elementary, high school, or college). Find the number of ways in which an individual can be classified.

  2. Eight elementary students are to be lined up to board a bus. How many different possible arrangements are there?

  3. In the Mathematics department of UW-L, there are 3 male and 2 female professors, 7 male and 2 female associate professors, and 4 male and 2 female assistant professors. A committee consisting of a professor, an associate professor, and an assistant professor is to be set up to review the current math curriculum of the department. Assuming that each faculty member has the same chance of being selected for the committee work, what is the probability that the committee will be composed of all female teachers?

- **Permutation.** Any ordered sequence of r objects taken from a set of n distinct objects is called a *permutation* of size r of the objects. [Theorem 2.2] The number of permutations of size r that can be constructed from n distinct objects is denoted by $P_r^n = P(n,r)$ and is given by

$$P_r^n = P(n,r) = n(n-1)(n-2)\cdots(n-r+2)(n-r+1) = \frac{n!}{(n-r)!}$$

  Examples:

  1. In a local swimming competition, there are 20 contestants. The first, second, and third placer will be awarded with gold, silver, and bronze medals, respectively. How many different possible competition results are there?

  2. Three girls and nine boys are to be lined up to get in a bus. In how many ways can this be done if all the girls insist to be together?

  3. In the previous example, suppose 2 of the boys (Mark and Jim) refused to be together. How many different arrangements are possible?

  4. How many different ways can you sit 12 students in a round table?

  5. How many different "words" can be made using all the letters of the word STATISTICS?

  6. How many different ways can you distribute 20 candies to 5 kids? What if you have to give at least one candy to each kid?

- **Combination.** Given a set of n distinct objects, any unordered subset of size r of the objects is called a *combination*. [Theorem 2.4] The number of combinations of size r that can be formed from n distinct objects will be denoted by $\binom{n}{r} = C(n, r)$, and is given by

$$\binom{n}{r} = C(n, r) = \frac{n!}{r!(n-r)!}$$

  Examples:

  1. If 2 cards are drawn from a well-shuffled standard deck of cards, in how many ways can you get two of the same suit?

  2. If 5 cards are drawn from a well-shuffled standard deck of cards, in how many ways can you get

  a. a flash (all of the same suit)

  b. a full house (a trio and a pair)

  3. There are 14 male and 6 female professors in the Mathematics department at UW-L. A committee of 5 members is to be formed to review the current math curriculum of the department. Find the number of ways this can be done if the committee must have at least 2 female professors.

- **Partitioning [Theorem 2.3].** The number of ways of partitioning n distinct objects into k distinct groups containing $n_!, n_2, \ldots, n_k$ objects, respectively, where each object appears in exactly one group and $n = \sum_{i=1}^{k} n_i$, is

$$N = \binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

  Examples:

  1. A labor dispute has arisen concerning the distribution of 20 laborers to four different construction jobs. The first job (considered to be very undesirable) required 6 laborers; the second, third, and fourth utilized 4, 5, and 5 laborers, respectively. The dispute arose over an alleged random distribution of the laborers to the jobs which placed all four members of a particular ethnic group on the first undesirable job. Determine the likelihood of this event occurring if the laborers were randomly assigned to the different jobs.

- **Recommended Problems.** Section 2.6: (pp. 48 - 51) #35, 37, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57, 61, 63, 69.