Presenting solution to a definite integral with substitution

Example: \[ \int_{3}^{5} \frac{2x}{(1 + x^2)^2} \, dx \]

**Solution 1 (preferred)**

\[
\text{indef: } \int \frac{2x}{(1 + x^2)^2} \, dx = \int \frac{1}{u^2} \, du = \int u^{-2} \, du = \frac{u^{-1}}{-1} + C = -\frac{1}{1 + x^2} + C
\]

\[
\begin{align*}
\text{substitution:} & \quad u = 1 + x^2 \\
\text{and } du = 2x \, dx
\end{align*}
\]

So, the definite integral is

\[
\left. \int_{3}^{5} \frac{2x}{(1 + x^2)^2} \, dx = \frac{u^{-1}}{-1} \right|_{3}^{5} = -\frac{1}{26} + \frac{1}{10} = \frac{16}{260} = \frac{4}{65}. \tag{⊕}
\]

- Usually least amount to write, especially if multiple substitutions or replacing back some \(x\)'s
- Least amount of ways to possibly make a mistake

**Solution 2**

\[
\int_{3}^{5} \frac{2x}{(1 + x^2)^2} \, dx = \int_{10}^{26} \frac{1}{u^2} \, du = \int_{10}^{26} u^{-2} \, du = \frac{u^{-1}}{-1} \bigg|_{10}^{26} = -\frac{1}{26} + \frac{1}{10} = \frac{16}{260} = \frac{4}{65}. \tag{⊕}
\]

\[
\begin{align*}
\text{substitution:} & \quad u = 1 + x^2 \\
\text{and } du = 2x \, dx \\
g(x) = 1 + x^2 \\
g(3) = 1 + 3^2 = 10 \\
g(5) = 1 + 5^2 = 26
\end{align*}
\]

- Often involves more writing
- On rare occasion, this is the more useful way to evaluate a definite integral

**Incorrect**

\[
\int_{3}^{5} \frac{2x}{(1 + x^2)^2} \, dx = \int_{3}^{5} \frac{1}{u^2} \, du = \int_{3}^{5} u^{-2} \, du = \frac{u^{-1}}{-1} \bigg|_{3}^{5} = -\frac{1}{1 + x^2} \bigg|_{3}^{5} = -\frac{1}{26} + \frac{1}{10} = \frac{16}{260} = \frac{4}{65}. \tag{⊕}
\]

- The second integral is **NOT** equal to the first because \(\int_{3}^{5} \frac{1}{u^2} \, du\) is the same as \(\int_{3}^{5} \frac{1}{2x} \, dx\).
- The problem is that the 3 and 5 are \(x\)-values but a “\(du\)” integral needs \(u\)-values as endpoints!
- The same problem affects the several expressions \(\int_{3}^{5} u^{-2} \, du\) and \(\frac{u^{-1}}{-1} \bigg|_{3}^{5}\) as well.

**Acceptable, but why??**

\[
\int_{3}^{5} \frac{2x}{(1 + x^2)^2} \, dx = \int_{3}^{5} \frac{1}{u^2} \, du = \int_{3}^{5} u^{-2} \, du = \frac{u^{-1}}{-1} \bigg|_{3}^{5} = -\frac{1}{1 + x^2} \bigg|_{3}^{5} = -\frac{1}{26} + \frac{1}{10} = \frac{16}{260} = \frac{4}{65}. \tag{⊕}
\]

- Problems fixed by specifying what’s an \(x\)-value when the expression only has \(u\)’s in it.
- Lots of room for errors (by forgetting the “\(x =\)” , which are time-consuming to write!)