1. Chapter 7 introduces new integral techniques and many will rely on using earlier techniques. Evaluate each integral using Chapter 5 methods.

(a) \( \int_{0}^{10} 2x \, dx \)
- **Solution outline:** Power rule antidifferentiation
- **2nd solution outline:** Area of a triangle

(b) \( \int (u + 4)(2u + 1) \, du \)
- **Solution outline:** FOIL, then Power rule

(c) \( \int \frac{1}{5\cos\theta \csc\theta} \, d\theta \)
- **Solution outline:** Think of the cosecant in the denominator as a sine in the numerator. Use \( u = \cos\theta \). Afterwards, use \( v = -u \).

(d) \( \int_{0}^{\sqrt{3}/2} \frac{1}{\sqrt{1-x^2}} \, dx \)
- **Solution outline:** this is \( \sin^{-1}(\sqrt{3}/2) - \sin^{-1}(0) = \frac{\pi}{3} - 0 \).

(e) \( \int (x + 2)e^{x^2+4x+17} \, dx \)
- **Solution outline:** Use \( u = x^2 + 4x + 17 \). The \( du \) will take care of \( x + 2 \), once 2's are multiplied in to the numerator and denominator.

(f) \( \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \tan x \, dx \)
- **Solution outline:** Use \( u = \cos x \).

(g) \( \int \frac{\sin \sqrt{x}}{\sqrt{x}} \, dx \)
- **Solution outline:** Use \( u = \sqrt{x} \).

(h) \( \int_{-8}^{8} \sqrt{64-y^2} \, dy \)
- **Solution outline:** Area of half a circle of radius 8.

(i) \( \int_{-8}^{8} \frac{1}{\sqrt{64-y^2}} \, dy \)
- **Solution outline:** Factor the 64, use \( u = \frac{1}{8}y \).

(j) \( \int \sin^4 x \cos x \, dx \)
- **Solution outline:** Use \( u = \sin x \).

(k) \( \int u^7 \sqrt{u^4 + 9} \, du \)
- **Solution outline:** Use \( v = u^4 + 9 \). Rewrite the \( u^7 \) as \( u^3 \cdot u^4 \). The \( u^4 \) part can get replaced with \( v - 9 \).
1. \( \int_{-\sqrt{3/3}}^{1} \frac{1}{1+3x^2} \, dx \)
   - **Solution outline:** Use \( u = \frac{1}{\sqrt{3}} x \).

2. \( \int \frac{x^4 + x^3 + x^2 + x + 1}{x^2} \, dx \)
   - **Solution outline:** Rewrite as the sum of 5 separate fractions. The first three terms will require Power Rule. The 4th term will antidifferentiate to natural log of absolute value of \( x \). The last term uses the Power Rule for integrals as well, but just has a negative exponent.
     Here's a little more detail:
     \[
     \int \frac{x^4 + x^3 + x^2 + x + 1}{x^2} \, dx = \int \left( x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} \right) \, dx \\
     = \int x^2 + x + 1 + \frac{1}{x} + x^{-2} \, dx \\
     = \frac{1}{3} x^3 + \frac{1}{2} x^2 + x + \ln|x| + \frac{1}{-1} x^{-1} + C \\
     = \frac{1}{3} x^3 + \frac{1}{2} x^2 + x + \ln|x| - \frac{1}{x} + C
     \]

3. \( \int \frac{\sec^2 x}{2 + \tan x} \, dx \)
   - **Solution outline:** Use \( u = 2 + \tan x \).

4. \( \int x^9 (x^5 + 3)^7 \, dx \)
   - **Solution outline:** Use \( u = x^5 + 3 \). Split the \( x^9 \) into two factors.

5. \( \int (\sin x + \cos x)^2 \, dx \)
   - **Solution outline:** FOIL first. Use the main Pythagorean identity. Then you can use \( u = \sin x \).
   - **2nd solution outline:** FOIL first. Use the main Pythagorean identity. Then you can use \( u = \cos x \).
   - **3rd solution outline:** FOIL first. Use the main Pythagorean identity. Then use the sine double angle identity. Then you can use \( u = 2x \).

2. If \( f(x) = 2e^x \csc x - (6 + \cot x)^{\sin^{-1} x} \), find \( f'(x) \)
   - **Solution:** We can consider one term at a time: in other words, by using
     \[
     f = 2e^x \csc x - (6 + \cot x)^{\sin^{-1} x}
     \]
     we have \( f = g - y \), thus \( f' = g' - y' \). The derivative of \( g \) is found by using the Product Rule. First, here's how I'd like to think of \( g \) as the product of two functions, though this is not the only way to do this:
     \[
     g = (2e^x) \csc x
     \]
     Then
     \[
     g' = (2e^x) \left( \csc x \right)' + (2e^x) \left( \csc x \right) = 2e^x (-\csc x \cot x) + 2e^x \csc x
     \]
     Now how about \( y' \)? Since \( y \) has \( x \)'s in BOTH the base AND exponent, the **ONLY** way to take the derivative is by logarithmic differentiation.
     \[
     y = (6 + \cot x)^{\sin^{-1} x}
     \]
\[
\ln y = \ln \left( (6 + \cot x)^{\sin^{-1} x} \right)
\]
\[
\ln y = \sin^{-1} x \cdot \ln(6 + \cot x)
\]

Now differentiate both sides with respect to \( x \), keeping in mind that \( y \) is a function of \( x \) on the left side:

\[
\frac{y'}{y} = \sin^{-1} x \cdot \frac{1}{6 + \cot x} \cdot (-\csc^2 x) + \frac{1}{\sqrt{1 - x^2}} \cdot \ln(6 + \cot x)
\]

By rewriting the algebra,

\[
\frac{y'}{y} = -\csc^2 x \cdot \sin^{-1} x \cdot \frac{1}{6 + \cot x} + \frac{\ln(6 + \cot x)}{\sqrt{1 - x^2}}
\]

Multiply by \( y \) on both sides:

\[
y' = y \left( -\csc^2 x \cdot \sin^{-1} x \cdot \frac{1}{6 + \cot x} + \frac{\ln(6 + \cot x)}{\sqrt{1 - x^2}} \right)
\]

Replace \( y \) with \((6 + \cot x)^{\sin^{-1} x}\) to get

\[
y' = (6 + \cot x)^{\sin^{-1} x} \left( -\csc^2 x \cdot \sin^{-1} x \cdot \frac{1}{6 + \cot x} + \frac{\ln(6 + \cot x)}{\sqrt{1 - x^2}} \right)
\]

Now, let's piece everything together:

\[
f''(x) = g'(x) - y'(x) = \frac{-2e^x \csc x \cot x + 2e^x \csc x}{g'(x)} - (6 + \cot x)^{\sin^{-1} x} \left( -\csc^2 x \cdot \sin^{-1} x \cdot \frac{1}{6 + \cot x} + \frac{\ln(6 + \cot x)}{\sqrt{1 - x^2}} \right)
\]

3. If \( y = \tan^{-1} x + \frac{5 \sin(x)}{3^x + \log_4 x} \), find \( \frac{dy}{dx} \)

- **Solution:** This is to remind you that in Leibnitz notation, the derivative of \( y \) with respect to input \( x \) is written as \( \frac{dy}{dx} \). The same thing in Newton notation is \( y' \) or \( y'(x) \). See the top half of page 157.

\[
\frac{dy}{dx} = \frac{1}{x^2 + 1} + \frac{(3^x + \log_4 x)(5 \cos x) - (5 \sin x)(3^x \ln 3 + \frac{1}{x \ln 4})}{(3^x + \log_4 x)^2}
\]

4. If \( y = \sec(x^4) + 3 \tan x \), find \( y' \)

- **Solution:** \( y' = \sec(x^4) \tan(x^4) \cdot 4x^3 + 3 \tan x \cdot 3 \cdot \sec^2 x \)

Be sure that you have the factor of \( \tan(x^4) \) in the first term and the fact that \( \tan(\ln x) \) is the use of \( x \) as the output and \( t \) as the input, as usually \( x \) is the input.

5. If \( x = t \sin t \), find \( \frac{d^2x}{dt^2} \)

- **Solution:** Writing \( \frac{dx}{dt} \) is the Leibnitz notation for the first derivative of \( x \) with respect to \( t \) and writing \( \frac{d^2x}{dt^2} \) is the second derivative of \( x \) with respect to \( t \). Read the top half of page 161 in the book for a refresher.

In Newton notation, \( \frac{dx}{dt} \) would be written \( x' \) and \( \frac{d^2x}{dt^2} \) would be written \( x'' \). What's unusual about this problem (but it's very common in later Calc classes) is the use of \( x \) as the output and \( t \) as the input, as usually \( x \) is the input.

By the Product Rule,

\[
\frac{dx}{dt} = t \cos t + \sin t
\]

To find the second derivative, we'll need the Product Rule on the first term:

\[
\frac{d^2x}{dt^2} = -t \sin t + \cos t + \cos t = 2 \cos t - t \sin t
\]

(a) \( \lim_{x \to \infty} \left( 1 + \frac{2}{3x} \right) \)
   • **Solution:** Use the Sum Law first. Then for the limit of \( \frac{2}{3x} \), this is a fixed number over a big number, whose limit is zero (see Theorem 5 in Section 2.6). The overall limit is 1.

(b) \( \lim_{x \to \infty} \left( 1 + \frac{2}{3x} \right)^x \)
   • **Solution:** Use L'Hospital's rule. (See the indeterminate forms at the end of Section 4.4.)

(c) \( \lim_{x \to 0} \frac{\sin 4x}{\sin 6x} \)
   • **Solution:** Use L'Hospital's Rule for this \( \frac{0}{0} \) indeterminate form.

(d) \( \lim_{x \to 0} \frac{\sin 4x}{\cos 6x} \)
   • **Solution:** Use the Quotient Law for limits (Section 2.3)

(e) \( \lim_{x \to 1} \frac{\ln x}{x - 1} \)
   • **Solution:** Use L'Hospital's Rule for \( \frac{0}{0} \) form.

(f) \( \lim_{x \to -\ln 3} e^x \)
   • **Solution:** Use the fact that \( e^x \) is continuous. Use the actual DEFINITION of continuity in Section 2.5.

(g) \( \lim_{x \to 0} \frac{\sin x}{x + \tan x} \)
   • **Solution:** Use L'Hospital's Rule for this \( \frac{0}{0} \) indeterminate form.

(h) \( \lim_{x \to 0} (\tan 2x)^x \)
   • **Solution:** Use L'Hospital's rule. (See the indeterminate forms at the end of Section 4.4.)

(i) \( \lim_{x \to 0} x \cot x \)
   • **Solution:** Rewrite this first as \( \lim_{x \to 0} \frac{x}{\tan x} \), which is a L'Hospital's \( \frac{0}{0} \) indeterminate form.

(j) \( \lim_{x \to \infty} \frac{1 - 6e^x}{1 + 13e^x} \)
   • **Solution:** Using L'Hospital's rule for this \( \frac{\infty}{\infty} \) form, you get
     \[
     \lim_{x \to \infty} \frac{1 - 6e^x}{1 + 13e^x} = \lim_{x \to \infty} \frac{-6e^x}{13e^x} = \lim_{x \to \infty} \frac{-6}{13} = -\frac{6}{13}
     \]

(k) \( \lim_{n \to \infty} n^{1/n} \)
   • **Solution:** Use L'Hospital's rule. (See the indeterminate forms at the end of Section 4.4.) If it helps you, first replace each \( n \) with an \( x \).
\[
\lim_{x \to \infty} \frac{8x^9 + 3x}{3x^{10} + x + 2}
\]

- **Solution:** One solution uses L'Hopital's Rule 9 times. Another option is to take the fraction and multiply by \( \frac{1}{x^{10}} \) in the numerator and the denominator. (See similar examples worked out in Section 2.6.)